Boundary Layers in Zero Pressure-Gradient Flows," *Journal of Applied Mechanics*, Vol. 39, 1972, pp. 25–32.

[4] Townsend, A. A., "Equilibrium Layers and Wall Turbulence," *Journal of Fluid Mechanics*, Vol. 11, 1961, pp. 97–116.

[5] Huffman, G. D. and Bradshaw, P., "A Note on Von Kármán's Constant in Low Reynolds Number Turbulent Flows," *Journal of Fluid Mechanics*, Vol. 53, 1972, pp. 45–60.

Reply by Author to P. Bradshaw and V. C. Patel

DAVID M. CHASE*
Bolt Beranek and Newman Inc., Cambridge, Mass.

BRADSHAW and Patel's comments are most cogent. The comparison of hypotheses for the mean velocity profile in turbulent axisymmetric flow along a cylinder given in Ref. 1, though correct, did not include the general hypothesis discussed by Bradshaw or, otherwise viewed, failed to define the most likely form of the "derivative" hypothesis.† It is this modified hypothesis cited by Bradshaw (hereafter called the local-similarity hypothesis), apart from inessential complications, that was applied to cylindrical flows in Refs. 2 and 3.

As defined in Ref. 1, the derivative hypothesis corresponded, in the logarithmic range, to a ratio of shear stress to eddy viscosity given by $(\tau/\rho)/\varepsilon = Av_*/y(1+y/a)$. Though, as stated by Bradshaw, this hypothesis corresponds to his Eq. (6) with $\tau = \tau_w/(1+y/a)^2$, it likewise corresponds—perhaps more sensibly though equally invalidly—to $\tau = \tau_w/(1+y/a)$ but with $\varepsilon = Ky(\tau_w/\rho)^{1/2}$ instead of $\varepsilon = Ky(\tau/\rho)^{1/2}$, i.e., with the eddy velocity scale assumed equal to its value at the wall instead of proportional to local shear-stress velocity. The local-similarity hypothesis, we note, implies an eddy viscosity given by $\varepsilon = A^{-1}v_*y(1+y/a)^{-1/2}$ in the logarithmic range, whereas Rao's hypothesis implies $\varepsilon = A^{-1}v_*a\ln(1+y/a)$.

Further quantitative comparison merits comment. The profile measured by Richmond[4] for $U = 460$ cm/sec with clay-centerbody trip and enveloping stovepipe, mentioned in Ref. 1, was especially smooth and the value of av_*/v small. A comparison like that of Fig. 1, Ref. 1, of the measured and variously computed profiles is given for this instance in Fig. 1 below with v_* adjusted in each computation to yield the measured result at a certain large y and with the planar profile taken as in Ref. 1, Eq. (7). (A comparison like that of Fig. 2, Ref. 1, is omitted but readily envisioned.) Rao's hypothesis is seen to yield excellent agreement with the measured profile even out to $y/a \sim 50$. As stated by Bradshaw, we have no right to expect any law-of-the-wall analysis to retain validity at such large y/a, and, furthermore, the assumption of local similarity finds success more generally than the argument for Rao's hypothesis. If accidental, the accuracy of the Rao profile is indeed remarkable.

A profile may be computed for the example of Fig. 1 also from the local-similarity hypothesis by use of Bradshaw's Eq. (7) for the logarithmic range with parameters v_* and C both adjustable. No profile so computed conforms to the measured one in Fig. 1 over such a large range of y/a (outside the sublayer) as does the Rao curve. Neither, however, does a Rao profile of the analogous approximate form

$$u/v_* = A\ln[(av_*/v)\ln(1+y/a)] + B$$

with v_* and B adjustable but without explicit treatment of the transitional profile. The Rao formulation retains the advantage of providing a simple explicit and successful prescription for incorporation of sublayer and transitional effects, e.g., by use of Eq. (7), Ref. 1.

References

[1] Chase, D. M., "Mean Velocity Profile of a Thick Turbulent Boundary Layer along a Circular Cylinder," *AIAA Journal*, Vol. 10, No. 6, June 1972, pp. 848–850.

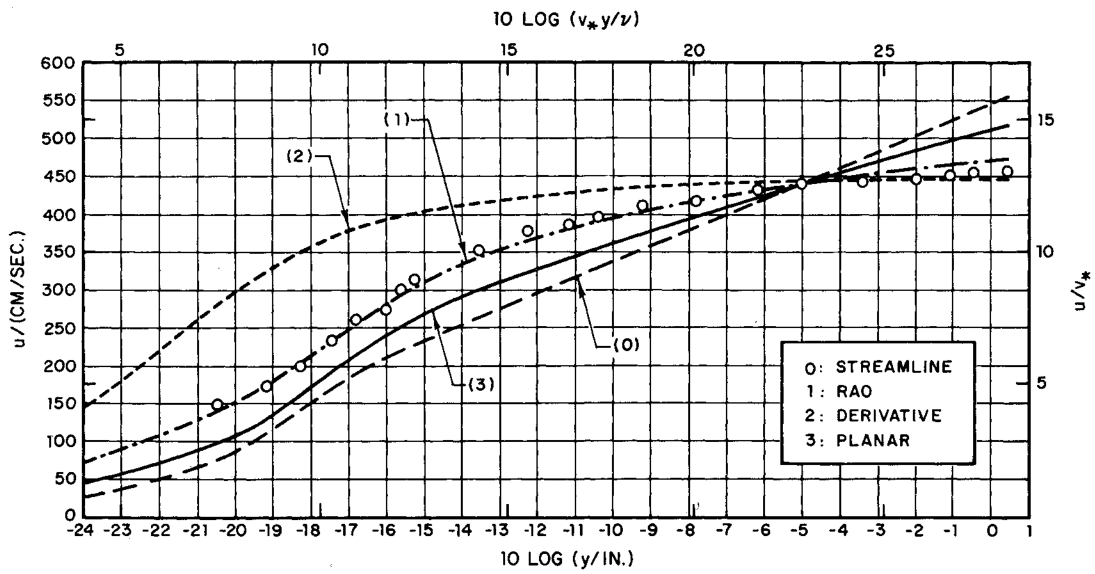

Fig. 1 Measured mean velocity profile for cylinder[4] and those computed from various hypotheses[1] with wall-friction velocity chosen in each case to yield measured velocity at $y = 0.315$ in. Dimensionless scales refer to $v_* = 35.2$ cm/sec of Rao curve.

Received December 15, 1972.
Index category: Boundary Layers and Convective Heat Transfer—Turbulent.

* Senior Scientist, Oceanology Department.

† The derivative hypothesis investigated in Ref. 1 was inadvertently stated incorrectly in Eq. (5). With $G(y_+, a_+) \equiv F(y_+, y_+/a_+)$, where F was defined in Eq. (1), Eq. (5) should be replaced by

$$\partial G(y_+, a_+)/\partial y_+ = (1+y_+/a_+)^{-1}\,dF(y_+, 0)/dy_+$$

Bradshaw, however, made the intended interpretation.

[2] Ginevskii, A. S. and Solodkin, E. E., "The Effect of Lateral Surface Curvature on the Characteristics of Axially-Symmetric Turbulent Boundary Layers," *Journal of Applied Mathematics and Mechanics*, Vol. 22, 1958, pp. 1169–1179.

[3] Sparrow, E., Eckert, E., and Minkowycz, W., "Heat Transfer and Skin Friction for Turbulent Boundary-Layer Flow Longitudinal to a Circular Cylinder," *Transactions of the ASME: Journal of Applied Mechanics*, Vol. 37, 1963, pp. 32–43.

[4] Richmond, R. L., "Experimental Investigation of Thick Axially Symmetric Boundary Layers on Cylinders at Subsonic and Hypersonic Speeds," Hypersonic Research Project Memo 39, June 1957, Guggenheim Aeronautical Lab., California Inst. of Technology, Pasadena, Calif.

Comment on "Laminar Thermal Boundary Layers on Continuous Surfaces"

H. E. EICKHOFF*
Institut für Überschalltechnik, Technische Universität, Berlin

IN a recent Note[1] heat-transfer rates in laminar boundary layers on continuous surfaces were discussed and compared to those of a semi-infinite flat plate. The analysis was performed numerically. As was shown in Ref. 2 this can be done analytically by an approximation yielding results which agree well with those obtained numerically. The calculation procedure is based on linearization of the boundary-layer equations, first proposed by Piercy and Preston[3] and Weyl.[4] Using a Taylor series expansion for the velocity at the wall results easily can be obtained as was shown by Schlünder[5] for the flat plate.

For constant wall temperature, the thermal boundary-layer equation in dimensionless form

$$(\partial^2\theta/\partial\eta)+\tfrac{1}{2}Pr(\partial\theta/\partial\eta)f=0 \tag{1}$$

with boundary values for dimensionless temperature $\theta_{\eta=0}=0$, $\theta_{\eta=\infty}=1$ can be integrated, when the dimensionless stream function f is assumed to be known. As solution for the heat-transfer coefficient one obtains

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0}=1\bigg/\int_0^\infty \exp\left(-\tfrac{1}{2}Pr\int_0^\eta f\,d\eta\right)d\eta \tag{2}$$

With two terms of a Taylor series expansion of the velocity profile $\partial f/\partial\eta$ at the wall

$$1+(\partial^2 f/\partial\eta^2)\eta+\cdots$$

Integration of Eq. (2) yields the final result

$$(\partial\theta/\partial\eta)_{\eta=0}\equiv Nu/(Re)^{1/2} \tag{3}$$
$$=(Pr)^{1/2}\bigg/\left[(\pi)^{1/2}-\frac{2}{3}\frac{(\partial^2 f/\partial\eta^2)_{\eta=0}}{(Pr)^{1/2}}+0.738\frac{(\partial^2 f/\partial\eta^2)_{\eta=0}}{Pr}\right.$$

With dimensionless wall-shear stress

$$(\partial^2 f/\partial\eta^2)_{\eta=0}=-0.444$$

results calculated from Eq. (3) are in rather good agreement with the curve shown in Fig. 1 of Ref. (1) in the range of

$$0.1\leq Pr\leq 1000$$

Even for more complex problems, as the boundary layer behind a shock wave with vaporization and combustion closed analytical solutions of suitable accuracy are obtained by linearization[2] compared to results of an analog computer study.[6]

The advantage of a closed solution is to show general tendencies. For example from Eq. (3) the asymptotic behavior with large Prandtl number may be seen. If $Pr\to\infty$, then $Nu/(Re\cdot Pr)^{1/2}=1/(\pi^{1/2})$. The same value is valid for the semi-infinite flat plate with $Pr\to 0$ explained by the fact, that in both cases flow velocity is a constant in the region of thermal boundary layer. If, however, $Pr\to 0$ in the case of the continuous moving surface then boundary-layer theory no longer is valid for heat-transfer calculations.

The potential flow induced by the boundary-layer flow is determined with the normal velocity component at the edge of boundary layer, which is $v_E\sim 1/x^{1/2}$. The same relation holds for the plane turbulent jet, where potential flow calculations already have been performed.[7] They show that the longitudinal velocity u_E is of the same order of magnitude as the normal component. Therefore heat-transfer calculations on the assumption of $v_E\sim 1/x^{1/2}$, $u_E=0$ yielding $Nu/(RePr)^{1/2}=0.808Pr^{1/2}$ [8] seem not to be correct. The same value of heat transfer is given in Ref. (1).

Received October 17, 1972.
Index categories: Boundary Layers and Convective Heat Transfer—Laminar; Shock Waves and Detonations; Hydrodynamics.
* Assistenz-Professor.

References

[1] Rhodes, C. A. and Kaminer, H., Jr., "Laminar Thermal Boundary Layers on Continuous Surfaces," *AIAA Journal*, Vol. 10, No. 3, March 1972, pp. 331–333.

[2] Eickhoff, H. E., "Näherungslösungen für laminare Grenzschichtprobleme bei kontinuierlich bewegter Wand," *Deutsche Luft- und Raumfahrt, Forschungsbericht* 72–26, Teil 1, 1972, p. 102.

[3] Piercy, N. A. V. and Preston, G. H., "A simple solution of the flat plate problem of skin friction and heat transfer," *Philosophical Magazine*, Vol. 7, No. 21, 1936, p. 996.

[4] Weyl, H., "Concerning the differential equations of some boundary layer problems," *Proceedings of the National Academy of Sciences*, Vol. 27, 1941, p. 578; also *Annual of Mathematics*, Vol. 43, 1942, p. 381.

[5] Schlünder, E. U., "Über analytische Näherungslösungen für laminare Grenzschichtprobleme," *Wärme- und Stoffübertragung*, Bd. 1, 1968, p. 35.

[6] Ragland, K. W., "Laminar Boundary Layer behind a Shock with Vaporization and Combustion," *AIAA Journal*, Vol. 8, No. 3, March 1970, p. 498.

[7] Kraemer, K., "Die Potentialströmung in der Umgebung von Freistrahlen," *Zeitschrift für Flugwissenschaften*, Vol. 19, 1972, Heft 3, p. 93.

[8] Rhodes, C. A., "Heat transfer in laminar magnetohydrodynamic boundary layers on continuous surfaces," *4th International Heat Transfer Conference*, Vol. II, Paris-Versailles, 1970.

Reply by Authors to H. E. Eickhoff

C. A. RHODES* AND H. KAMINER JR.†
University of South Carolina, Columbia, S.C.

THE authors wish to thank H. E. Eickhoff for informing us of his work on continuous moving surfaces and for his interesting comments. The series expansion technique used by Eickhoff has the advantage that it leads to a better understanding of the asymptotic behavior with some sacrifice in accuracy.

Eickhoff correctly points out we have assumed in Ref. 1 that the freestream longitudinal velocity component $u_E=0$ negligible in determining the small Prandtl number heat transfer. The energy equation for the region outside the boundary layer contains the terms $u_E\,\partial T/\partial x$ and $v_E\,\partial T/\partial y$. Since the temperature gradient in the x direction is small compared to that in the y direction, the former term is small compared to the latter even

Received December 26, 1972.
Index categories: Boundary Layers and Convective Heat Transfer—Laminar; Shock Waves and Detonation; Hydrodynamics.
* Associate Professor. Member AIAA.
† Undergraduate Student.